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Non-polynomial spline method for the solution of the dissipative wave equation

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Abstract

Purpose – The purpose of this paper is to propose a non-polynomial spline-based method to obtain numerical solutions of a dissipative wave equation. Applying the Von Neumann stability analysis, the developed method is shown to be conditionally stable for given values of specified parameters. A numerical example is given to illustrate the applicability and the accuracy of the proposed method. The obtained numerical results reveal that our proposed method maintains good accuracy.

Design/methodology/approach – A non-polynomial spline is proposed based on the dissipative wave equation, which gives nonlinear system of algebraic equations; by solving these equations, the numerical solution is found.

Findings – It is found that the method gives more accurate numerical results for such nonlinear partial differential equations. The stability is good.

Research limitations/implications – Any nonlinear or linear partial differential equation can be solved by such method.

Practical implications – We compare between the numerical and analytic solutions of the dissipative wave equation, also the error norms which were small.

Originality/value – This paper presents a new method to solve such problems.

Keywords Differential equations, Numerical analysis, Polynomials, Stability (control theory)

Paper type Research paper

1. Introduction

In this paper we propose a non-polynomial spline-based method to obtain numerical solutions of the dissipative wave equation of the form (Adomian, 1994):

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + 2u_t u = g(x,t) \quad (1)$$

subject to the conditions:

$$u(a,t) = \eta_1, \quad u(b,t) = \eta_2, \quad t \geq 0 \quad (2)$$

and:

$$u(x,0) = f_1(x), \; u_t(x,0) = f_2(x) \quad a \leq x \leq b \quad (3)$$

Recently, there is a wide use to the non-polynomial spline-based methods for approximating the solution of boundary value problems of different orders (see for example Daele *et al.*, 1994; Islam *et al.*, 2005; Ramadan *et al.*, 2007, 2008). However, the numerical analysis literature contains little for using these non-polynomial splines

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dealing with numerical solutions of partial differential equations (El-Danaf and Abd Alaal, 2006; Rahidinia, 2007; Ramadan *et al.*, 2007).

The spline functions proposed have the form $T_3 = span\{1, x, \sin \omega x, \cos \omega x\}$ where ω is the frequency of the trigonometric part of the spline functions which will be used to raise the accuracy of the method.

This paper is organized as follows: In section 2, a new method depends on the use of the non-polynomial splines is derived. In section 3, the stability analysis is theoretically discussed. Using Von Neumann method, for given values of specified parameters, the proposed method is shown to be conditionally stable. Finally, in section 4 a numerical example is included to illustrate the practical implementation of the proposed method.

2. Derivation of the numerical method

To set up the non-polynomial spline method, select an integer $N > 0$ and time-step size $k > 0$. With $h = \frac{b-a}{N+1}$, the mesh points (x_i, t_j) are:

$$x_i = a + ih, \text{for each } i = 0, 1, \ldots, N+1,$$

and,

$$t_j = jk, \text{ for each } j = 0, 1, \ldots$$

Let $Z_i^j \equiv Z(x_i, t_j)$ be an approximation to $u(x_i, t_j)$, obtained by the segment $P_i(x, t_j)$ of the mixed spline function passing through the points (x_i, Z_i^j) and (x_{i+1}, Z_{i+1}^j). Each segment has the form:

$$P_i(x, t_j) = a_i(t_j) \cos \omega(x - x_i) + b_i(t_j) \sin \omega(x - x_i) + c_i(t_j)(x - x_i) + d_i(t_j) \tag{4}$$

For each $i = 0, 1, \ldots, N$ we define:

$$\begin{aligned} &P_i(x_i, t_j) = Z_i^j, \quad P_i(x_{i+1}, t_j) = Z_{i+1}^j, \quad P_i^{(2)}(x_i, t_j) = S_i^j, \quad \text{and} \\ &P_i^{(2)}(x_{i+1}, t_j) = S_{i+1}^j, \end{aligned} \tag{5}$$

where $P_i^{(2)}(x_i, t_j) \equiv \frac{\partial^2}{\partial x^2} P_i(x_i, t_j)$.

Using Equations (4) and (5), we obtain expressions for the coefficients of (4) in terms of Z_i^j, Z_{i+1}^j, S_i^j, and S_i^{j+1} as:

$$\begin{aligned} &a_i + d_i = Z_i^j, \\ &a_i \cos \theta + b_i \sin \theta + c_i h + d_i = Z_{i+1}^j \\ &-a_i \omega^2 = S_i^j \\ &-a_i \omega^2 \cos \theta - b_i \omega^2 \sin \theta = S_{i+1}^j \end{aligned} \tag{6}$$

where $a_i \equiv a_i(t_j),\ b_i \equiv b_i(t_j),\ c_i \equiv c_i(t_j),\ d_i \equiv d_i(t_j)$, and $\theta = \omega h$.

Solving the last four equations, we obtain the following expressions:

$$a_i = -\frac{h^2}{\theta^2} S_i^j, \; b_i = \frac{h^2(\cos\theta\, S_i^j - S_{i+1}^j)}{\theta^2 \sin\theta}, \; c_i = \frac{(Z_{i+1}^j - Z_i^j)}{h} + \frac{h(S_{i+1}^j - S_i^j)}{\theta^2}$$
$$d_i = \frac{h^2}{\theta^2} S_i^j + Z_i^j, \qquad (7)$$

2.1 Spline relations

Using the continuity condition of the first derivative at $x = x_i$, that is $P_i^{(1)}(x_i, t_j) = P_{i-1}^{(1)}(x_i, t_j)$, we obtain:

$$b_i\omega + c_i = -a_{i-1}\omega \sin\theta + b_{i-1}\omega \cos\theta + c_{i-1} \qquad (8)$$

Using equation (7), equation (8) becomes:

$$\frac{h^2\omega(\cos\theta\, S_i^j - S_{i+1}^j)}{\theta^2 \sin\theta} + \frac{(Z_{i+1}^j - Z_i^j)}{h} + \frac{h(S_{i+1}^j - S_i^j)}{\theta^2} =$$
$$= \frac{h^2\omega}{\theta^2} S_{i-1}^j \sin\theta + \frac{h^2\omega(\cos\theta\, S_{i-1}^j - S_i^j)}{\theta^2 \sin\theta}\cos\theta + \frac{(Z_i^j - Z_{i-1}^j)}{h} + \frac{h(S_i^j - S_{i-1}^j)}{\theta^2}$$

After slight rearrangements, the last equation reduces to:

$$Z_{i+1}^j - 2Z_i^j + Z_{i-1}^j = \alpha S_{i+1}^j + \beta S_i^j + \alpha S_{i-1}^j, \quad i = 1, 2, \ldots, N. \qquad (9)$$

where $\alpha = \frac{h^2}{\theta \sin\theta} - \frac{h^2}{\theta^2}$, and $\beta = -\frac{2h^2 \cos\theta}{\theta \sin\theta} + \frac{2h^2}{\theta^2}$.

Remark. As $\omega \to 0$, that is $\theta \to 0$,then $(\alpha, \beta) \to \left(\frac{h^2}{6}, \frac{4h^2}{6}\right)$, and system (9) reduces to ordinary cubic spline:

$$Z_{i+1}^j - 2Z_i^j + Z_{i-1}^j = \frac{h^2}{6}(S_{i+1}^j + 4S_i^j + S_{i-1}^j), \quad i = 1, 2, \ldots, N.$$

Using differential equation (1), we can write S_i^j in the form:

$$S_i^j = \frac{\partial^2 Z_i^j}{\partial x^2} = \left(\frac{\partial^2 Z_i^j}{\partial t^2} + \delta_i^j Z_i^j - g_i^j\right) \qquad (10)$$

where $\delta_i^j = 2\frac{\partial Z_i^j}{\partial t}$.

Using Equation (10), S_{i+1}^j,S_i^j, and S_{i-1}^j can be discretized as follows:

$$
\begin{aligned}
S_{i+1}^j &= \left(\frac{Z_{i+1}^{j-1} - 2Z_{i+1}^j + Z_{i+1}^{j+1}}{k^2} + \delta_{i+1}^j Z_{i+1}^j - g_{i+1}^j \right) \\
S_i^j &= \left(\frac{Z_i^{j-1} - 2Z_i^j + Z_i^{j+1}}{k^2} + \delta_i^j Z_i^j - g_i^j \right) \\
S_{i-1}^j &= \left(\frac{Z_{i-1}^{j-1} - 2Z_{i-1}^j + Z_{i-1}^{j+1}}{k^2} + \delta_{i-1}^j Z_{i-1}^j - g_{i-1}^j \right)
\end{aligned}
\tag{11}
$$

Inserting forms in (11) for S_{i+1}^j,S_i^j, and S_{i-1}^j into Equation (9), we obtain:

$$
\begin{aligned}
Z_{i+1}^j - 2Z_i^j + Z_{i-1}^j = \alpha &\left(\frac{Z_{i-1}^{j-1} - 2Z_{i-1}^j + Z_{i-1}^{j+1}}{k^2} + \delta_{i-1}^j Z_{i-1}^j - g_{i-1}^j \right) \\
&+ \beta \left(\frac{Z_i^{j-1} - 2Z_i^j + Z_i^{j+1}}{k^2} + \delta_i^j Z_i^j - g_i^j \right) \\
&+ \alpha \left(\frac{Z_{i+1}^{j-1} - 2Z_{i+1}^j + Z_{i+1}^{j+1}}{k^2} + \delta_{i+1}^j Z_{i+1}^j - g_{i+1}^j \right)
\end{aligned}
\tag{12}
$$

After simple calculations, the above equation becomes:

$$
\begin{aligned}
\alpha Z_{i-1}^{j+1} + \beta Z_i^{j+1} + \alpha Z_{i+1}^{j+1} &= (k^2 + 2\alpha - \alpha k^2 \delta_{i-1}^j) Z_{i-1}^j \\
&+ (-2\,k^2 + 2\beta - \beta k^2 \delta_i^j) Z_i^j \\
&+ (k^2 + 2\alpha - \alpha k^2 \delta_{i+1}^j) Z_{i+1}^j - \alpha Z_{i-1}^{j-1} - \beta Z_i^{j-1} \\
&- \alpha Z_{i+1}^{j-1} + \lambda_i^j, \quad i = 1, 2, \ldots, N.
\end{aligned}
\tag{13}
$$

where $\lambda_i^j = k^2(\alpha g_{i-1}^j + \beta g_i^j + \alpha g_{i+1}^j)$ and $\delta_i^j = 2\frac{\partial Z_i^j}{\partial t} \approx \frac{2(Z_i^j - Z_i^{j-1})}{k}$. System (13) consists of N equations in the $N+2$ unknowns $Z_i,\ i = 0, \ldots, N+1$. To get a solution to this system we need two additional equations. These equations are obtained from the boundary conditions in (2). The two parts in (2) are replaced by:

$$
\begin{aligned}
Z_0^j &= \eta_1 \\
Z_{N+1}^j &= \eta_2, \quad j = 0, 1, \ldots
\end{aligned}
\tag{14}
$$

Writing Equations (13) and (14) in matrix form gives:

$$
A\,Z^{j+1} = B\,Z^j - C\,Z^{j-1} + r \tag{15}
$$

where,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \alpha & \beta & \alpha & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \alpha & \beta & \alpha & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & & & \dots \\ 0 & & & & & & & & 0 \\ 0 & \dots & 0 & 0 & 0 & \alpha & \beta & \alpha & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & \alpha & \beta & \alpha \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ A_1 & B_1 & C_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & A_2 & B_2 & C_2 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & & & \dots \\ 0 & & & & & & & & 0 \\ 0 & \dots & 0 & 0 & 0 & A_{N-1} & B_{N-1} & C_{N-1} & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & A_N & B_N & C_N \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \alpha & \beta & \alpha & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \alpha & \beta & \alpha & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & & & \dots \\ 0 & & & & & & & & 0 \\ 0 & \dots & 0 & 0 & 0 & \alpha & \beta & \alpha & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & \alpha & \beta & \alpha \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$r = (\eta_1, \lambda_1, \lambda_2, \dots, \lambda_N, \eta_2)^T$$

and

$$A_i = k^2 + 2\alpha - k^2\alpha\delta_{i-1}^j$$
$$B_i = -2k^2 + 2\beta - k^2\beta\delta_i^j$$
$$C_i = k^2 + 2\alpha - k^2\alpha\delta_{i+1}^j$$

Equations (13) and (14) imply that the $(j+1)$th time-step requires values from the (j)th and $(j-1)$th time steps. This produces a minor starting problem since values for $j = 0$ *are* given by the first part in Equation (3):

$$Z_i^0 = u(x_i, 0) = f_1(x_i), \quad i = 1, \dots, N. \tag{16}$$

but values for $j = 1$, which are needed in Equation (15) to compute Z_i^2, must be obtained from the second part in (3):

$$\frac{\partial Z_i^0}{\partial t} = u_t(x_i, 0) = f_2(x_i), \quad i = 1, \ldots, N.$$

One approach is to replace $\partial Z_i^0/\partial t$ by a forward-difference approximation:

$$f_2(x_i) = \frac{\partial Z_i^0}{\partial t} = \frac{Z_i^1 - Z_i^0}{k} + o(k) \tag{17}$$

which gives us:

$$Z_i^1 \approx Z_i^0 + kf_2(x_i), \quad i = 1, \ldots, N. \tag{18}$$

The last result gives an approximation that has local truncation error of only $O(k)$. A better approximation to Z_i^1 can be obtained rather easily, particularly when the second derivative of $u(x, 0) = f_1$ at x_i can be determined. Using the Taylor polynomial up to the first two terms in t for Z at $(x_i, 0)$ we can write:

$$\frac{Z_i^1 - Z_i^0}{k} = \frac{\partial Z_i^0}{\partial t} + \frac{k}{2}\frac{\partial^2 Z_i^0}{\partial t^2} + o(k^2) \tag{19}$$

Suppose Equation (1) holds on the initial line; that is:

$$\frac{\partial^2 u}{\partial t^2}(x_i, 0) - \frac{\partial^2 u}{\partial x^2}(x_i, 0) + 2u_t(x_i, 0)u(x_i, 0) = g(x_i, 0), \quad \text{for each } i = 0, 1, \ldots N+1.$$

If $f_1^{(2)}$ exists, then:

$$\frac{\partial^2 Z_i^0}{\partial t^2} = \frac{\partial^2 u}{\partial t^2}(x_i, 0) = g_i^0 + \frac{\partial^2 u}{\partial x^2}(x_i, 0) - 2u_t(x_i, 0)u(x_i, 0) = g_i^0 + \frac{d^2 f_1}{dx^2}(x_i) - 2f_2(x_i)f_1(x_i),$$

Substituting into Equation (19) and solving for Z_i^1 gives:

$$Z_i^1 \approx Z_i^0 + kf_2(x_i) + \frac{k^2}{2}\left(g_i^0 + \frac{d^2 f_1}{dx^2}(x_i) - 2f_2 f_1\right), \quad i = 1, \ldots, N. \tag{20}$$

This is an approximation with local truncation error of $O(k^2)$.

3. The stability analysis

To investigate stability of the scheme, we apply the Von Neumann method after linearizing the nonlinear difference equation (13) by taking $\delta_{i+1}, \delta_i,$ *and* δ_{i-1} as a local constant d^*. According to the Von Neumann method we have:

$$Z_i^j = \zeta^j \exp(q\varphi ih), \tag{21}$$

where φ is the wave number, $q = \sqrt{-1}$, h is the element size, and ζ is the amplification factor. The use of Equations (21) and (13) gives us the characteristic equation in the form

$$\zeta^{j+1}\{\alpha\,\exp((i-1)q\varphi h)+\beta\,\exp(iq\varphi h)+\alpha\,\exp((i+1)q\varphi h)\}=$$
$$\zeta^{j}\left\{\begin{array}{l}(k^2+2\alpha-\alpha k^2d^*)\exp((i-1)q\varphi h)+(-2k^2+2\beta-\beta k^2d^*)\exp(iq\varphi h)+\\(k^2+2\alpha-\alpha k^2d^*)\exp((i+1)q\varphi h)\end{array}\right\}-$$
$$\zeta^{j-1}\{\alpha\,\exp((i-1)q\varphi h)+\beta\,\exp(iq\varphi h)+\alpha\,\exp((i+1)q\varphi h)\}$$

Dividing both sides of the last equation by $\exp(iq\varphi h)$ we obtain:

$$\begin{array}{l}\zeta^{j+1}\{\alpha\,\exp(-q\varphi h)+\beta+\alpha\,\exp(q\varphi h)\}=\\ \zeta^{j}\left\{\begin{array}{l}(k^2+2\alpha-\alpha k^2d^*)\exp(-q\varphi h)+(-2k^2+2\beta-\beta k^2d^*)+\\(k^2+2\alpha-\alpha k^2d^*)\exp(q\varphi h)\end{array}\right\}-\\ \zeta^{j-1}\{\alpha\,\exp(-q\varphi h)+\beta+\alpha\,\exp(q\varphi h)\}\end{array}\tag{22}$$

After canceling the common term, that is $\zeta^{j-1}\{\alpha\,\exp(-q\varphi h)+\beta+\alpha\,\exp(q\varphi h)\}$, Equation (22) becomes:

$$\varsigma^2\;+\;2\mu\varsigma\;+\;1\;=\;0\tag{23}$$

where

$$\mu=\frac{(\alpha k^2d^*-k^2)\exp(-q\phi)+(\beta k^2d^*+2k^2)+(\alpha k^2d^*-k^2)\exp(q\phi)}{2(\alpha\,\exp(-q\phi)+\beta+\alpha\,\exp(q\phi))}-1,\text{ and }\phi=\varphi h$$

or

$$\mu=\frac{2(\alpha k^2d^*-k^2)\cos\,\phi+(\beta k^2d^*+2k^2)}{2(2\alpha\,\cos\,\phi+\beta)}-1\tag{24}$$

Equation (23) is a quadratic in ζ and hence will have two roots, that is $\zeta_{\pm}=-\mu\pm\sqrt{\mu^2-1}$. For stability, we must have $|\zeta_{\pm}|\leq 1$. Also from Equation (23) we can observe that the product of the two values of ζ is clearly unity. So three cases arise.

Case 1: Both the roots are equal to unity. In that case the discriminant of the quadratic equation (23) is zero.

Case 2: One of the roots is greater than unity. In that case the discriminant is greater than zero. This means that stability condition, that is $|\zeta_{\pm}|\leq 1$, is not satisfied. In other words, ζ^j would grow in an unbounded manner.

Case 3: Discriminant is less than zero, that is: $\mu^2-1<0$.

Thus, for stability:

$$-1\leq\mu\leq 1\tag{25}$$

Using Equation (24), the above inequality becomes:

$$-\frac{k^2d^*}{2}\leq\frac{k^2(1-\cos\,\phi)}{(\beta+2\alpha\,\cos\,\phi)}\leq 2-\frac{k^2d^*}{2}$$
$$-\frac{k^2d^*}{2}\leq\frac{2k^2\,\sin^2(\phi/2)}{(\beta+2\alpha)-4\alpha\,\sin^2(\phi/2)}\leq 2-\frac{k^2d^*}{2}\tag{26}$$

Two cases will be discussed:

Case 1: For $\beta = -2\alpha$, inequality (26) becomes:

$$-\frac{k^2 d^*}{2} \leq \frac{k^2}{-2\alpha} \leq 2 - \frac{k^2 d^*}{2} \tag{27}$$

The right inequality in (27) which can be written in the form:

$$\frac{k^2}{-2\alpha} \leq 2 - \frac{k^2 d^*}{2} \tag{28}$$

is satisfied for $\alpha < 0,\ k^2 \ll |\alpha|,\ and\ k^2$ small enough to make:

$$\left(2 - \frac{k^2 d^*}{2}\right) \rightarrow 2 \text{ and } 0 < \frac{k^2}{-2\alpha} \ll 1$$

but the left inequality, that is $(-d^*/2) \leq (1/-2\alpha)$, is valid for $|\alpha|$ small enough and $\alpha < 0$ to make $(1/-2\alpha) > 0$. Finally, we can say that our system is stable for $\beta = -2\alpha, \alpha < 0,$ and $k^2 \ll |\alpha|$ such that $|\alpha|$, and k^2 are small enough.

Case 2: For $\alpha > 0, \beta > 2\alpha$, the quantity $(\beta + 2\alpha) - 4\alpha\ \sin^2(\phi/2)$ is positive, so the right inequality in (26) which can be written in the form:

$$2k^2\ \sin^2(\phi/2) \leq \left(2 - \frac{k^2 d^*}{2}\right)(\beta + 2\alpha - 4\alpha\ \sin^2(\phi/2)) \tag{29}$$

is satisfied for $\alpha > 0, \beta > 0, \beta \gg 2\alpha$, and $k^2 \ll \beta$ small enough to make $\left(2 - (k^2 d^*/2)\right) \rightarrow 2$ and $2k^2\ \sin^2(\phi/2) \rightarrow 0$, but the left inequality, that is:

$$-d^* \leq \frac{4\ \sin^2(\phi/2)}{(\beta + 2\alpha) - 4\alpha\ \sin^2(\phi/2)} \tag{30}$$

is valid for $\alpha > 0, \beta > 0$, and $\beta > 2\alpha$ such that α, and β are small enough and sin $(\phi/2) \neq 0$. Finally, we can say that stability in this case requires $\alpha > 0$, $\beta > 0$, and $\beta > 2\alpha$ such that α, β and $k^2 \ll \beta$ are small enough and $\sin(\phi/2) \neq 0$.

4. Numerical results

We now obtain the numerical solution of dissipative wave equation for one standard problem. The accuracy of our proposed numerical method is measured by computing the difference between the analytic and numerical solutions at each mesh point and use these to compute the maximum absolute error, $L_\infty - error$ norm.

Example. Consider the dissipative wave equation (Adomian, 1994):

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + 2u_t u = -2\ \sin^2\ x\ \sin\ t\ \cos\ t\ 0 \leq x \leq \pi, \quad t \geq 0 \tag{31}$$

with the initial conditions:

$$u(x,0) = \sin\ x \quad u_t(x,0) = 0, \tag{32}$$

and the boundary conditions:

$$u(0,t) = \ u(\pi,t) = 0, \tag{33}$$

The exact solution of this problem is:

$$u(x,t) = \sin x \cos t \tag{34}$$

From the obtained numerical results in Tables I-V, we can conclude that applying non-polynomial splines in the solution of partial differential equations is a promising approach

Table I.

Time	0.500	1.500	2.500	3.000
$L_\infty - error$	2.55058×10^{-4}	1.76001×10^{-3}	3.21326×10^{-3}	4.25842×10^{-3}

Notes: $h = \pi/40$; $k = 0.01$; $\alpha = -1.01$; $\beta = -2\alpha$

Table II.

Time	0.500	1.500	2.500	3.000
$L_\infty - error$	2.52751×10^{-4}	1.73065×10^{-3}	4.31818×10^{-3}	4.9997×10^{-3}

Notes: $h = \pi/40$; $k = 0.01$; $\alpha = 10^{-5}$; $\beta = 6.21 \times 10^{-3}$

Table III.

Time	0.500	1.500	2.500	3.000
$L_\infty - error$	4.90515×10^{-5}	2.75733×10^{-4}	6.19493×10^{-4}	7.59812×10^{-4}

Notes: $h = \pi/40$; $k = 0.0025$; $\alpha = 2 \times 10^{-7}$; $\beta = 6.18 \times 10^{-3}$

Table IV.

x	Exact solution	Numerical solution
0.2π	0.04157828392871431	0.041764024258423134
0.3π	0.05722759828369953	0.057475017088467340
0.4π	0.06727507659055325	0.067547692975227380
0.5π	0.07073720166770290	0.071012934346015150
0.6π	0.06727507659055325	0.067547692975153300
0.7π	0.05722759828369953	0.057475017088395690
0.8π	0.04157828392871431	0.041764024258327160

Notes: $h = \pi/40$; $t = 1.5$; $\alpha = 2 \times 10^{-7}$; $\beta = 6.18 \times 10^{-3}$

Table V.

x	Exact solution	Numerical solution
0.2π	−0.47090040218675855	−0.47064203333751690
0.3π	−0.64813879991245870	−0.64771280914059780
0.4π	−0.76193285605417060	−0.76136720198110770
0.5π	−0.80114361554693370	−0.80052412271783310
0.6π	−0.76193285605417060	−0.76136720198119200
0.7π	−0.64813879991245870	−0.64771280914063270
0.8π	−0.47090040218675855	−0.47064203333747706

Notes: $h = \pi/40$; $t = 2.5$; $\alpha = 2 \times 10^{-7}$; $\beta = 6.18 \times 10^{-3}$

5. Conclusion

In this paper a numerical treatment for a dissipative wave equation using non-polynomial spline is proposed. Applying the Von Neumann stability analysis, the developed method is shown to be conditionally stable for given values of specified parameters. The obtained numerical results show that our proposed method maintains good accuracy.

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